BRANCHING OF CRACKS PRODUCED IN A PLATE BY IMPACT

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The problem of crack propagation in the presence of a large number of branching events is considered in the case of a semi-infinite inhomogeneous plate to whose edge an impact load is applied. The theory of random processes is employed. The mean-square distance and the mean-square angular deviation of the cracks from the branching axis are determined; these quantities are convenient for comparison with the experimental data.

1. Exact solutions of the dynamic equations of the theory of elasticity [1,2] have shown that in an isotropic elastic-brittle plate branching of a crack is possible at a propagation velocity exceeding $0.6c_t$ (transverse elastic wave velocity). The angle between the expected direction of branching and the direction of propagation of a crack moving at a velocity of $0.8c_t$ is equal to 30-60° [1]. Experimental studies [1,3,4] have revealed a large number of crack branching events, in which two equivalent cracks are formed, in connection with the brittle fracture of plates composed of various materials. In these studies the first crack branched whenever its propagation velocity exceeded a certain value. The angle between the first and the newly formed crack did not exceed 20-40°.

Below, the theory of random processes [5] is employed to determine the mean-square distance and the mean-square angular deviation of the cracks from the branching axis, which coincides with the direction of propagation of the first crack. A large number of crack branching events is investigated in a semi-infinite inhomogeneous elastic-brittle plate subjected to the action of an impact load (Fig. 1).

2. Real materials consist of a large number of "primary" elements: grains, crystals, etc. Within each small element the material is homogeneous, but relative to each other the elements are inhomogeneous with respect to mechanical properties or the state of stress and strain. Such a material we shall call "quasi-homogeneous." Let an inhomogeneity consist of quasi-homogeneous material and contain a considerable number of primary elements. Then by an "inhomogeneous" medium we understand a quasi-homogeneous material in which the inhomogeneities are statistically distributed.

We will define the concept of crack energy E. The energy balance equation for crack propagation [6], referred to the entire length of the crack, has the form

$$W = P + T \tag{2.1}$$

Here, W is the liberated elastic energy, $P = 2\gamma s$ is the surface energy, and $T = k\rho l^2 v^2 \sigma^2 / 2E^2$ is the kinetic energy of the crack (usual notation in the expressions for P and T). By the energy of a propagating crack, completely determined by the physicomechanical characteristics and the state of stress of the material, we understand the sum of the surface and kinetic energies. In this case it is possible to give an explanation of crack branching in energy terms. Crack propagation leads to an increase in the liberated elastic energy. As the fracture rate increases, the kinetic energy rises to a certain limit determined by the maximum possible crack propagation velocity. In accordance with Eq. (2.1) the surface energy must also increase, i.e., the fracture surface must become larger, which also explains the formation of new cracks at the tip of the moving crack.

Thus, the branching process occurs when the crack energy reaches a certain value. At a sufficiently large crack propagation velocity the kinetic energy obviously makes the principal contribution to the energy

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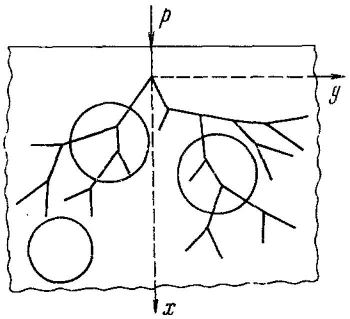

Fig. 1

of the crack. Accordingly, the above conclusion does not contradict the previously obtained theoretical [1,2] and experimental [3,4] results concerning the effect of the propagation velocity on crack branching.

We formulate the following hypotheses:

1) Any crack with an energy exceeding a certain value branches into two secondary cracks.

2) Crack branching occurs in the quasi-homogeneous material and in the inhomogeneities of the plate, the infinitely thin interface not being a source of cracks.

3) The angular deviation of the cracks associated with branching is small.

We will obtain a vector equation for the probability $a(\mathbf{R}, \mathbf{y}, x)d\mathbf{R}^*dy$ of finding a crack within the limits $\mathbf{R}, \mathbf{R} + d\mathbf{R}$ at a distance x from the edge of the plate and with horizontal deviation $\mathbf{y}, \mathbf{y} + dy$ from the branching axis, which coincides with the direction of propagation of the first crack. In this case

$$d\mathbf{R}^* = \frac{dR_x dR_y}{k_1}$$

where $\mathbf{R}$ is the vector analog of the crack energy E, and k_1 is the crack shape parameter.

Let $b(\mathbf{R}_1, \mathbf{R})d\mathbf{R}^*$ determine the probability of a crack with $\mathbf{R}, \mathbf{R} + d\mathbf{R}$ being produced from a crack with $\mathbf{R}_1$; $c(x)$ is the density of the inhomogeneities in the plane; then $k_2c(x)dx/\cos\alpha$ is the probability of hitting an inhomogeneity in the layer dx. Here, α is the angle between the vertical and the direction of crack propagation, k_2 is a constant for a given plate material.

We will consider a crack with $\mathbf{R}$, whose tip is located at a distance x from the edge of the plate. The probability that a crack in the layer $h - x$ (above x) will not hit an inhomogeneity is equal to $\exp\{[L(h) - L(x)]/\cos\alpha]\}$, where

$$L(x) = \int_x^\infty c(x_1)\, dx_1.$$

The probability of this crack emerging from an inhomogeneity located in the layer h, h + dh, is $L_1(h)dh/\cos\alpha$. The probability that the preceding crack had $\mathbf{R}_1, \mathbf{R}_1 + d\mathbf{R}_1$ is equal to $a(\mathbf{R}_1, \mathbf{r}, h)d\mathbf{R}_1^*d\mathbf{r}$, where $\mathbf{r}, \mathbf{r} + d\mathbf{r}$ − ee is its displacement from the branching axis of the cracks.

The equation for the crack distribution function a has the form

$$a(\mathbf{R}, \mathbf{y}, x) = -\int_x^\infty \int a(\mathbf{R}_1, \mathbf{r}, h)\, b(\mathbf{R}_1, \mathbf{R}) \exp\frac{L(h) - L(x)}{\cos\alpha}$$
$$\times L_1(h)\, dh / \cos\alpha\, d\mathbf{R}_1^* + a(\mathbf{R}, \mathbf{r}_\infty, \infty) \exp\frac{-L(x)}{\cos\alpha} \tag{2.2}$$

where

$$r = y - R_y(h - x)/R_x, \quad \mathbf{r} = \mathbf{r}_\infty \text{ at } h = \infty$$

We differentiate (2.2) with respect to x and y

$$\frac{\partial a}{\partial L}\cos\alpha - \frac{R_y}{RL_1}\frac{\partial a}{\partial y} + a = \int a(\mathbf{R}_1)\, b(\mathbf{R}_1, \mathbf{R})\, d\mathbf{R}_1^* \tag{2.3}$$

Here, L_1 is a function of L. For a plate in which the density of the inhomogeneities is constant $L_1 = -k_2c$; in this case the unit of length can be so selected that $k_2c = 1$.

We perform on a a Fourier transformation with respect to y

$$\psi(\mathbf{R}, \rho, L) = \int a(\mathbf{R}, \mathbf{y}, L) \exp(-i\rho\mathbf{y})\, dy \tag{2.4}$$

Then

$$\psi = \sum_m \psi_m(\mathbf{R}, L)\, \rho^m / m! \tag{2.5}$$

where

$$\psi_m = \int (-iy)^m a(\mathbf{R}, \mathbf{y}, L)\, dy \tag{2.6}$$

From (2.3) we obtain the equation for ψ

$$\frac{\partial\psi}{\partial L}\cos\alpha - \frac{i\rho\cdot\mathbf{R}}{L_1 R}\psi + \psi = \int \psi(\mathbf{R}_1)\, b(\mathbf{R}_1, \mathbf{R})\, d\mathbf{R}_1{}^* \tag{2.7}$$

We differentiate (2.7) m times with respect to ρ and set $\rho = 0$. We write the obtained equations for ψ_m in the previous variables

$$\frac{\partial x(\mathbf{R})}{\partial L}\cos\alpha + x(\mathbf{R}) = \int x(\mathbf{R}_1)\, b(\mathbf{R}_1, \mathbf{R})\, d\mathbf{R}_1{}^* \tag{2.8}$$

$$\frac{\partial y(\mathbf{R})}{\partial L}\cos\alpha + y(\mathbf{R}) = \int y(\mathbf{R}_1)\, b(\mathbf{R}_1, \mathbf{R})\, d\mathbf{R}_1{}^* \tag{2.9}$$

We employ the notation: β is the angle between $\mathbf{R}$ and $\mathbf{R}_1$, β_1 is the angle between $\mathbf{R}_1$ and the vertical. Then

$$\cos\beta_1 = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Equations (2.8) and (2.9) take the form

$$\frac{\partial x(E, \cos\alpha)}{\partial L}\cos\alpha + x(E, \cos\alpha) = \int_E^\infty \int_{-1}^{+1} x(E_1, \cos\beta_1)\, b(E_1, E, \cos\beta)\, dE_1\, d(\cos\beta) \tag{2.10}$$

$$\frac{\partial y(E, \cos\alpha)}{\partial L}\cos\alpha + y(E, \cos\alpha) = \int_E^\infty \int_{-1}^{+1} y(E_1, \cos\beta_1)\, b(E_1, E, \cos\beta)\, dE_1\, d(\cos\beta) \tag{2.11}$$

3. We will determine the mean-square distance and the mean-square angular deviation of the cracks from an axis coinciding with the direction of propagation of the first crack. These quantities take the form

$$\langle d^2(E, L)\rangle = \left[\int_{-1}^{+1} x(E, \cos\alpha, L)\, d(\cos\alpha)\right]\left[\int_{-1}^{+1} y(E, \cos\alpha, L)\, d(\cos\alpha)\right]^{-1} \tag{3.1}$$

$$\langle f^2(E, L)\rangle = \left[\int_{-1}^{+1} y(E, \cos\alpha, L)\sin^2\alpha\, d(\cos\alpha)\right]\left[\int_{-1}^{+1} y(E, \cos\alpha, L)\, d(\cos\alpha)\right]^{-1} \tag{3.2}$$

In order to find $\langle d^2\rangle$ and $\langle f^2\rangle$ it is necessary to determine the coefficients of the expansion of $x(E, \cos\alpha)$ and $y(E, \cos\alpha)$ in Legendre polynomials.

$$x(E, \cos\alpha) = \sum_{K=0}^{\infty}\left(K + \frac{1}{2}\right) x_K(E)\, P_K(\cos\alpha) \tag{3.3}$$

$$y(E, \cos\alpha) = \sum_{K=0}^{\infty}\left(K + \frac{1}{2}\right) y_K(E)\, P_K(\cos\alpha) \tag{3.4}$$

Then the unknown quantities take the form

$$\langle d^2(E)\rangle = x_0(E)/y_0(E) \tag{3.5}$$

$$\langle f^2(E)\rangle = 2/3\,[y_0(E) - y_2(E)]/y_0(E) \tag{3.6}$$

We substitute (3.3) and (3.4) in Eqs. (2.10) and (2.11), respectively, and simplify. Applying the addition theorem for Legendre polynomials

$$P_K(\cos\beta_1) = P_K(\cos\alpha)\, P_K(\cos\beta) \tag{3.7}$$

provided that

$$b_K(E_1, E) = \int_{-1}^{+1} b(E_1, E, \cos\beta)\, P_K(\cos\beta)\, d(\cos\beta) \tag{3.8}$$

we find

$$\int_{-1}^{+1} x(E_1, \cos\beta_1)\, b(E_1, E, \cos\beta)\, d(\cos\beta) = \sum_{K=0}^{\infty} \left(K + \frac{1}{2}\right) x_K(E_1)\; b_K(E_1, E)\; P_K(\cos\alpha) \tag{3.9}$$

Multiplying (2.10) by $P_K(\cos\alpha)$ and integrating with respect to $\cos\alpha$ from -1 to $+1$, we obtain

$$\frac{\partial x_1(E)}{\partial L} + x_0(E) = \int_E^{\infty} x_0(E_1)\, b_0(E_1, E)\, dE_1 \tag{3.10}$$

But $x(E, \cos\alpha)$ is proportional to $\sin^2\alpha$; accordingly, on the basis of hypothesis 3) we can neglect the expression

$$x_1(E) - x_0(E) = \int_{-1}^{+1} x(E, \cos\alpha)(\cos\alpha - 1)\, d(\cos\alpha) \tag{3.11}$$

i.e., replace $x_1(E)$ in (3.10) by $x_0(E)$.

Solving the equation obtained by means of a Mellin transformation, we obtain

$$x_0(E, L) = \frac{2v}{E_0^2} \int_{s_0 - i\infty}^{s_0 + i\infty} \left(\frac{E}{E_0}\right)^{-(s+1)} j(s)\, Q(s, L)\, ds \tag{3.12}$$

where

$$v = (2\pi i)^{-1}$$

$$Q(s, L) = \frac{L \exp[-t(s) L]}{t(s-1) - t(s)} - \frac{1}{2} L^2 \exp[-t(s) L] \tag{3.13}$$

It can be shown that the mean number of cracks at a distance L in the plate is given by the expression

$$y_0(E, L) = \frac{v}{E_0} \int_{s_0 - i\infty}^{s_0 + i\infty} \left(\frac{E}{E_0}\right)^{-(s+1)} \exp[-t(s) L]\, ds \tag{3.14}$$

where

$$t(s) = 1 - \int_0^{\infty} \left(\frac{E}{E_1}\right)^s E_1 b_0(E_1, E)\, d\left(\frac{E}{E_1}\right) \tag{3.15}$$

if branching began from a single crack with energy E_0.

Then the mean-square distance of the cracks from the branching axis, coinciding with the direction of propagation of the first crack, is equal, in accordance with expression (3.5), to the ratio of (3.12) to (3.14).

Multiplying (2.11) by $P_K(\cos\alpha)$ and integrating with respect to $\cos\alpha$ from -1 to $+1$, we obtain

$$\frac{\partial}{\partial L}[H_1 y_{K+1}(E) + H_2 y_{K-1}(E)] + y_K(E) = \int\limits_E^\infty y_K(E_1)\, b_K(E_1, E)\, dE_1 \tag{3.16}$$

where

$$H_1 = \frac{K+1}{2K+1}, \quad H_2 = \frac{K}{2K+1}$$

By virtue of hypothesis 3) we neglect terms depending on the coefficient $(1 - \cos\alpha)^2$, which varies as the mean of the fourth power of α.

Substituting K = 0, K = 1 in Eq. (3.16) and subtracting the results, we find

$$\frac{\partial}{\partial L}[y_0(E) - y_2(E)] + [y_0(E) - y_2(E)] = \int\limits_E^\infty [y_0(E_1) - y_2(E_1)]\, b_0(E_1, E)\, dE_1$$
$$+ \int\limits_E^\infty y_0(E_1)\, [b_0(E_1, E) - b_2(E_1, E)]\, dE_1 \tag{3.17}$$

The solution of this equation, obtained by means of a Mellin transformation, has the form

$$\frac{2}{3}[y_0(E, L) - y_2(E, L)] = \frac{v}{E_0^2} \int\limits_{s_0 - i\infty}^{s_0 + i\infty} \left(\frac{E}{E_0}\right)^{-(s+1)} j(s)\, G(s)\, ds \tag{3.18}$$

where

$$G(s) = \exp[-t(s-1)L] - \exp[-t(s)L] \tag{3.19}$$

$$[t(s) - t(s-1)]\, j(s) = \frac{2}{3}\int\limits_0^\infty \left(\frac{E}{E_1}\right)^s E_1^2\, [b_0(E_1, E) - b_2(E_1, E)]\, d\left(\frac{E}{E_1}\right) \tag{3.20}$$

It can be shown that $b_0(E_1, E)$ and $b_0(E_1, E) - b_2(E_1, E)$ have the form

$$b_0(E_1, E) = \frac{v}{E_1} \int\limits_{s_0 - i\infty}^{s_0 + i\infty} \left(\frac{E}{E_1}\right)^{-(s+1)} (1 - q_1)\, ds \tag{3.21}$$

$$\frac{2}{3}[b_0(E_1, E) - b_2(E_1, E)] = \frac{v}{E_1^2} \int\limits_{s_0 - i\infty}^{s_0 + i\infty} \left(\frac{E}{E_1}\right)^{-(s+1)} (q_1 - q_2)\, n(s)\, ds \tag{3.22}$$

where

$$q_1 = q[Nl(s)], \quad q_2 = q[Nl(s-1)]$$

$$q(\sigma) = 1 - 2[1 - (1 + \sigma)]\, e^{-\sigma}]\, \sigma^{-2} \tag{3.23}$$

$$n(s) = [2 - l(s) - l(s-1)]\, [l(s) - l(s-1)]^{-1} \tag{3.24}$$

$$l(s) = 1 - 2\int\limits_0^\infty \int\limits_0^\infty \mu^s \lambda(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 \tag{3.25}$$

N is the mean number of branchings that a crack undergoes in passing through an inhomogeneity, $\lambda(\mu_1, \mu_2)$ is the probability of the energies of the cracks formed being in the ratios μ_1, μ_2 to the energy of the first crack.

From Eqs. (3.15) and (3.21) there follows

$$t(s) = q\,[Nl(s)] \tag{3.26}$$

and from (3.20) and (3.22)

$$j(s) = n(s) \tag{3.27}$$

Thus, the mean-square angular deviation from an axis coinciding with the direction of propagation of the first crack is equal, in accordance with expression (3.6), to the ratio of (3.18) to (3.14).

4. In order to obtain results we write the initial condition for the number of cracks at a distance L from the edge of the plate

$$y_0(E,\ L = 0) = \delta(E - E_0) \tag{4.1}$$

$$Y_0(E,\ L = 0) = 1 \tag{4.2}$$

The expression for $y_0(E, L)$ is given in (3.14) and

$$Y_0(E, L) = \frac{1}{2\pi i}\int\limits_{s_0 - i\infty}^{s_0 + i\infty}\left(\frac{E}{E_0}\right)^{-s}\exp[-t(s)L]\,\frac{ds}{s} \tag{4.3}$$

where t(s) is given by (3.26), (3.25), and (3.23).

The mean-square distance of the cracks from the branching axis has the form

$$\langle d^2(E, L)\rangle = \frac{2v}{y_0(E, L)\,E_0^2}\int\limits_{s_0 - i\infty}^{s_0 + i\infty}\left(\frac{E}{E_0}\right)^{-(s+1)} j(s)\,Q(s, L)\,ds \tag{4.4}$$

$$\langle D^2(E, L)\rangle = \frac{2v}{Y_0(E, L)}\,\frac{1}{E_0}\int\limits_{s_0 - i\infty}^{s_0 + i\infty}\left(\frac{E}{E_0}\right)^{-s} j(s)\,Q(s, L)\,\frac{ds}{s} \tag{4.5}$$

where the values of v, j(s) and Q(s, L) were determined in Sec. 3.

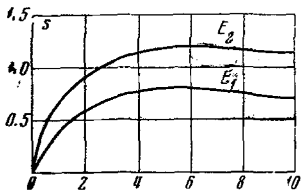

Fig. 2

Thus, the mean-square distance from the branching axis of the cracks formed in a plate as a result of impact is inversely proportional to the energy of the first crack and after first increasing begins to decrease as a result of the departure of the cracks from the energy interval in question.

For the mean-square angular deviation of the cracks from the branching axis we have

$$\langle f^2(E, L)\rangle = \frac{v}{y_0(E, L)\,E_0^2}\int\limits_{s_0 - i\infty}^{s_0 + i\infty}\left(\frac{E}{E_0}\right)^{-(s+1)} j(s)\,G(s)\,ds \tag{4.6}$$

$$\langle F^2(E, L)\rangle = \frac{v}{Y_0(E, L)}\,\frac{1}{E_0}\int\limits_{s_0 - i\infty}^{s_0 + i\infty}\left(\frac{E}{E_0}\right)^{-s} j(s)\,G(s)\,\frac{ds}{s} \tag{4.7}$$

where G(s) is given by (3.19).

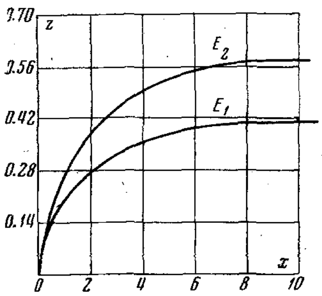

Fig. 3

Thus, the mean-square angular deviation of the cracks from a branching axis coinciding with the direction of propagation of the first crack in a semi-infinite elastic-brittle inhomogeneous plate is inversely proportional to the energy of the first crack and after initially increasing tends to a certain constant value.

In Figs. 2 and 3 we have plotted in relative units the dependence on the length of the plate of the quantities s and z, which are respectively equal to the square roots of the mean-square distance and the mean-square angular deviation of the cracks from the branching axis in a semi-infinite inhomogeneous plate, whose edge is subjected to the action of a unit impact load. In this case branching begins from a single crack with energy E_0.

Comparison with the crack branching data for sheets of glass obtained by Shardin and co-workers [3] shows that the results of the proposed theory (increase of the mean-square distance and the mean-square angular deviation of the cracks from the branching axis in a quasi-homogeneous plate) are in qualitative agreement with the experimental data.

In [3] the procedure for loading the test specimens was as follows. The sheets of glass were first loaded in tension. Then fracture was initiated by the impact of a knife edge against the edge of the sheet. In this case the mean-square distance and the mean-square angular deviation from a branching axis coinciding with the direction of propagation of the first crack continuously increased. Obviously, this was because the energy of the cracks formed increased at the expense of the elastic energy accumulated in the specimen-machine system.

Thus, it is not possible to expect complete qualitative agreement between the experimental and theoretical data (for the particular case of a quasi-homogeneous plate) in view of the difference between the assumed plate loading procedure and that employed in [3].

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